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RESEARCH ARTICLE

Comparative Analysis and Definitions of Fractional Derivatives

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Abstract

Fractional Calculus (FC) has emerged as a valuable tool in various fields. This study explores the historical development of (FC) and examines prominent definitions regarding Fractional Derivatives (FD), such as the Riemann-Liouville, Grunwald-Letnikov, Caputo Fractional Derivative, Katugampula derivatives, Caputo Fractional Derivative, Caputo-Fabrizio Fractional Derivative and as well as Atangana-Baleanu Fractional Derivative. It critically evaluates their strengths, weaknesses and implications on (FD) equations. The findings contribute to establishing a clearer understanding of Fractional Derivatives (FD) and guiding their appropriate use in practical scenarios. It investigates the impact of different definitions on the properties and behaviors of (FD), providing valuable insights for researchers.

Introduction

Fractional calculus

The study and application of arbitrary order integrals and derivatives is the focus of Fractional Calculus (FC), a branch of mathematical analysis [1]. Since the FC, non-integer order notion in calculus, has gained popularity over the past four to five decades, is having a significant impact on all the traditional domains of mathematics, physics, chemistry, engineering and economics [2].

Fractional derivatives

A useful technique for describing memory phenomena is Fractional Derivative (FD). Although it does not represent any physical process, the memory function is the kernel function of FD. Ambiguity about the physical meaning has been a significant barrier keeping FD far behind integer order calculus. What are the physical meanings of FC? Was proposed as an open subject in 1974.

By using analogous reasoning, a physical explanation based on inhomogeneous and changing time scales was presented in 2002; however, no experiments have been conducted to support the new time scale. The open issue continues to have no obvious solution yet. It was discovered that the fractional order can be used to suit the test data for memory phenomena from many areas [2].

History of fractional derivative

L'Hospital wrote Leibniz a letter in 1695. A crucial query regarding the

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order of the derivative that arose from his message was, what could a derivative of order 1/2 be? Leibniz's prescient response predicts the emergence of the area that is today known as FC. In fact, FC dates the classical calculus, which was independently developed by Newton and Leibniz.

Contrary to what happens in the case of FC, the derivative in classical calculus has an important geometric interpretation in that it is connected to the idea of a tangent. The slow development of FC up to 1900 can be considered as a drawback because of this difference. Following Leibniz, Euler was the first to identify the issue with a derivative of non-integer order [3].

Gamma function

The definition of FD depends solely on the gamma function. The gamma function discovered by Euler in 1729, which is defined as [4].

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}\,dt, where\ \Re(z)>0,$$

Application of fractional derivative

FD has applications in different fields including control theory, biology and electroanalytical chemistry, physics and economics.

1.5.1. Control theory: Outstaloup invented the so-called commande robusted'ordre non entire controller, which is discussed in a series of his books on applications of FD in control theory and is the founder of the concept of employing fractional order controllers for the control of dynamical systems [5]. In contrast to the proportional integral derivative controller, Outstaloup showed the commande robusted'ordre non entire controller's advantage. In comparison to the traditional proportional integral derivative controller, the proportional integral derivative controller performs better when used to control fractional order systems [6].

In addition to providing some very interesting concepts for applying FD to control theory, the works of Bagley and Calico, Matignon, and D'Andrea-Novel also offer some approaches for researching fractional order control systems [5]. Recently, boundary controls for integer order infinite dimensional systems have been implemented using fractional integrals and FDs, studied by Montseny, Audounet, Matignon, as well as Mbodje and Montseny. The employment of FDs and fractional integrals in control theory produces better results than integer order approaches, and it also serves as a powerful incentive for furthering the generalization of established research techniques and the interpretation of findings [4]. A fractional-order vector space depiction, which is an extension of the state space notion, is described, together with initiation response and control techniques for vector visualizations of established fractional-order structures [5].

Definitions of fractional derivative

A derivative of arbitrary order is first mentioned in print in 1819. Less than two pages of the 700-page work on differential and integral calculus written by the French mathematician Lacroix were devoted to this subject.

Beginning with,

$$y=x^n,$$

by power rule,

$$\frac{dy}{dx}=nx^{n-1},$$

$$\frac{d^2y}{dx^2}=\frac{n!}{(n-2)!}x^{n-2},$$

similarly,

$$\frac{d^3y}{dx^3}=\frac{n!}{(n-3)!}x^{n-3},$$

for any $\alpha<n$,

$$\frac{d^\alpha y}{dx^\alpha}=\frac{n!}{(n-\alpha)!}x^{n-\alpha}, \tag{1}$$

α is not necessarily an integer but the factorial function in the denominator cannot exist [7]. In 1730, Euler defined the gamma function and modified the Eq. (1), [8].

$$D^\alpha y=\frac{n!}{\tilde{A}(n-\alpha+1)}x^{n-\alpha}.$$

Riemann liouville fractional derivative: The fractional order η Riemann-Liouville derivative (RLFD) of the function $f(t)$ is given in [4].

$$D_x^{\eta} f(x) = \frac{1}{\tilde{A}(n-\eta)} \frac{d^n}{dx^n} \int_a^x (x-t)^{n-\eta-1} f(t) dt.$$

Grunwald-letnikov fractional derivative: In 1867 Letnikov and Grünwald defined FD of order η of [4]. $f(t)$ is given as,

$$f^{\eta}(x) = \lim_{h \to 0} \frac{1}{h^n} \sum_{j=0}^{\frac{t-a}{h}} \frac{\Gamma(\eta+1)}{j! \tilde{A}(\eta-j+1)} (-1)^j \binom{\eta}{j} f(x-mh).$$

1.1.2.Caputo fractional derivative: The majority of RLFD's shortcomings were overcome by Caputo in 1967, when he made the most significant contributions to FC [9].

The Caputo FD (CFD) of order η of $f(t)$ is given as,

$$D_x^{\eta} f(x) = \frac{1}{\tilde{A}(n-\eta)} \int_a^x (x-t)^{n-\eta-1} \left[\frac{d^n f(t)}{dt^n} \right] dt.$$

Katugampola fractional derivative: The Katugampola FD used in [10] can be written as,

Let $f(x)$ be continuous function $(0,\infty)$,

$\eta \in (0,1), p > -1,$

and,

$\forall\, 0 < a < t.$

Then,

$$({}_x^p D_{a+}^{\eta} f)(x) = \frac{(p)^{\eta-n+1}}{\tilde{A}(n-\eta)} \left(x^{1-p} \frac{d}{dx} \right)^n \int_a^x \frac{t^{p-1} f(t)}{(x^p - t^p)^{\eta-n+1}} dt.$$

1.1.4.Caputo fabrizio fractional derivative: Caputo–Fabrizio introduced a definition in 2015, which have non-singular kernel which is defined as [11],

$$D_t^{\eta}(f(t)) = \frac{M(\eta)}{1-\eta} \int_a^t f'(x) e^{\left[-\eta \frac{t-x}{1-\eta}\right]} dx,$$

where,

$f \in H^1(a,b), b > a, \alpha \in [0,1],$

and,

$M(0) = M(1) = 1.$

and $M(\eta)$ is a normalization function. However, the derivative can be redefined as follows if $f \notin H^1(a,b)$,

$$D_t^{\eta}(f(t)) = \eta \frac{M(\eta)}{1-\eta} \int_a^t (f(t) - f(x)) e^{\left[-\eta \frac{t-x}{1-\eta}\right]} dx.$$

Atangana-baleanu fractional derivative: In 2016, Atangana-Baleanu introduced Atangana-Baleanu FD (ABFD). There are two types of ABFD.

Atangana-Baleanu Fractional Derivative in Caputo sense: The ABFD in Caputo sense is given,

$$ {}^{ABC}_{a}D_t^{\alpha}[f(t)] = \frac{B(\alpha)}{1-\alpha} \int_a^t f'(x) E_{\alpha} \left[-\alpha \frac{(t-x)^{\alpha}}{1-\alpha} \right] dx. \qquad (2)$$

where,

$f \in H^1(a,b), a < b, \alpha \in [0,1].$

Naturally, B(α) has the same characteristic as in the case of Caputo and Fabrizio. The aforementioned definition is useful for discussing practical issues and is also very advantageous for applying the Laplace transform to some physical issues with initial conditions.

Atangana-Baleanu Fractional Derivative in Riemann–Liouville sense: The ABFD in RLFD sense is defined as,

$$ {}^{ABR}_{a}D_t^{\alpha}[f(t)] = \frac{B(\alpha)}{1-\alpha} \frac{d}{dt} \int_a^t f(x) E_{\alpha} \left[-\alpha \frac{(t-x)^{\alpha}}{1-\alpha} \right] dx, \qquad (3)$$

where,

$f \in H^1(a,b), a < b, \alpha \in [0,1],$

The kernel in Eq. (2) and Eq. (3) is non-local. Also, in Eq. (2), when the function is constant, we get zero.

where, $M(\alpha) > 0$ is a normalization function with $M(0) = M(1) = 1$ [12].

Coimbra fractional derivative: Coimbra FD with variable order of η is given as follows (Moghadam, 2020):

$$D_t^{\eta} f(x,t) = \frac{1}{\tilde{A}(1-\eta)} \int_{0+}^t (t-\varsigma)^{\eta} \frac{\partial f(x,\varsigma)}{\partial \varsigma} d\varsigma + \frac{(f(x,0^+) - f(x,0^-)) t^{-\eta}}{\tilde{A}(1-\eta)},$$

where,

$0 < \eta \le 1.$

Suppose that $f \in C^2(\Omega)$. We will survey the solution $f(x,t)$ for $t \geq 0$. The operator would correspond to the CFD in the case $f(x,0^+) = f(x,0^-)$ in the Coimbra FD. As a result, it is presumed that this property when $f(x,t)$ in $t = 0$ is good enough. As a result, in place of Coimbra derivative, the term Caputo derivative can be used with assumption that the attribute is set in $t = 0$.

Advantages of fractional derivative

There are different advantages of the definitions of FDs which are discussed in detail.

Advantages of Caputo Fractional Derivative: The CFD has the following advantages [13]:

- It enables the problem formulation to include traditional initial and boundary conditions

It obeys the property of classical derivative

$$\frac{d}{dx}constant = 0$$

Derivative does not vanish at origin

The CFD of constant is 0

Advantages of caputo- fabrizio fractional derivative: The advantages of Caputo- Fabrizio FD are listed as:

- It has a non-singularity in its kernel
- The boundary conditions of the Caputo-Fabrizio derivatives based fractional differential equations permit the same form as the differential equations of integer order
- A constant's Caputo-Fabrizio derivative is zero [14]

Advantages of atangana-baleanu fractional derivative: The ABFD has the following advantages:

- It has non-singular and non-local kernel
- It is used as the generalized Mittag-Leffler function [15]

Disadvantages of fractional derivative

Disadvantages of Lacroix Fractional Derivative

In this section, we discuss the disadvantages of all the definitions of FDs.

The Lacroix derivative has some drawbacks

$$\frac{d^\eta y}{dx^\eta} = \frac{n!}{(n-\eta)!}x^{n-\eta}.$$

This equation is derived by Leibniz and L'Hospital. This formula fails because the η is not necessarily an integer due to which the factorial function does not exist [11].

Disadvantages of euler fractional derivative: The Euler FD has some drawbacks

$$D^\eta y = \frac{n!}{\Gamma(n-\eta+1)!}x^{n-\eta}$$

is useful but it violates one of the basic property of the classical derivative.

It violates the property of $\frac{d}{dx}(constant) = 0$

in the case of this formula the differentiation of the constant function does not exist

This formula is also not applicable to the exponential function

Disadvantages of the riemann–liouville fractional derivative: The RLFD has some drawbacks

- FD of constant is not zero in RLFD
- Not applicable when $f(x)$ is not differentiable

Violates $\frac{d}{dx}constant = 0$

Derivative vanish at the origin [13]

Disadvantages of the grunwald-letnikov fractional derivative: The Grunwald-Letnikov derivative has some drawbacks

- The Grunwald-Letnikov FD is defined only when limit exist

Disadvantages of the caputo fractional derivative: The CFD has some drawbacks

- To calculate the FD of a function by CFD, we must first compute its classical derivative, which imposes stricter requirements on differentiability and regularity
- It is defined only for differentiable functions [13]

Disadvantages of the caputo–fabrizio fractional derivative: The Caputo–Fabrizio has some drawbacks

There have been various challenges of the Caputo-Fabrizio derivative, though.

- The Caputo-Fabrizio derivative uses a local kernel
- Instead of fractional integral, the anti-

derivative of their derivative is the average of the function and its integral

- We do not recover the starting function when the fractional order is zero [13].

Disadvantages of the atangana-baleanu fractional derivative: There are several issues with the ABFD in the Caputo meaning.

Unless the function vanishes at the origin, we cannot recover the original function when $\eta = 0$ [12]

Conclusion and Future Recommendations

From the above discussion it is clear that a lot of research work will be done on fractional derivatives in the present. It is a wide research area that investigated the impact of different definitions on the properties and behaviors of fractional derivatives along with the advantages and disadvantages of each derivative type. This provides a valuable insight for researchers in the coming future study [16].

References

1. Mainardi F, Gorenflo R. Fractional calculus and special functions. Phys. Univ. Bologna. 2013;1-64.
2. Du M, Wang Z, Hu H. Measuring memory with the order of fractional derivative. Scientific Reports. 2013;3(1):1-3.
3. De Oliveira EC, Tenreiro Machado JA. A review of definitions for fractional derivatives and integral. Mathematical Problems in Engineering. 2014;1-6.
4. Dalir M, Bashour M. Applications of fractional calculus. Applied Mathematical Sciences. 2010;4(21):1021-1032.
5. Hartley TT, Lorenzo CF. Dynamics and control of initialized fractional-order systems. Nonlinear Dynamics. 2002;29:201-233.
6. You H, Shen Y, Xing H, Yang S. Optimal control and parameters design for the fractional-order vehicle suspension system. Journal of Low Frequency Noise, Vibration and Active Control. 2018;37(3):456-467. doi: 10.1177/0263092317717166.
7. Mladenov V, Mastorakis N. Advanced topics on applications of fractional calculus on control problems, system stability and modeling. World Scientific and Engineering Academy and Society. 2014;1-12.
8. Miller KG, Ross B. An Introduction to the Fractional Calculus and Fractional Differential Equations.1st ed. New York:Wiley; 1993:1-384
9. Khaliq A, Mujeeb R. Existence of weak solutions for Ψ-Caputo fractional boundary value problem via variational methods. J. Appl. Anal. Comput. 2021;11:768-1778. doi: 10.11948/20200115.
10. Nazeer N, Asjad MI, Azam MK, Akgül A. Study of Results of Katugampola Fractional Derivative and Chebyshev Inequailities. International Journal of Applied and Computational Mathematics. 2022; 8(5):1-16. doi: 10.1007/s40819-022-01426-x.
11. Moore EJ, Sirisubtawee S, Koonprasert S. A Caputo–Fabrizio fractional differential equation model for HIV/AIDS with treatment compartment. Advances in Difference Equations. 2019;(1):1-20. doi: 10.1186/s13662-019-2138-9.
12. Bas E, Ozarslan R. Real world applications of fractional models by Atangana–Baleanu fractional derivative. Chaos, Solitons & Fractals. 2018; 116:121-125. doi: 10.1016/j.chaos.2018.09.019.
13. Atangana A. Fractional Operators and Their Applications. In Elsevier eBooks. 2018;79–112. doi: 10.1016/B978-0-12-809670-3.00005-9.
14. Bouzenna FEG, Meftah MT, Difallah M. Application of the Caputo Fabrizio derivative without singular kernel to fractional Schrödinger equations. Pramana. 2020;94(1):1-7. doi: 10.1007/s12043-020-01968-z.
15. Atangana A, Koca I. Chaos in a simple nonlinear system with Atangana–Baleanu derivatives with fractional order. Chaos, Solitons & Fractals. 2016; 89:447-454. doi: 10.1016/j.chaos.2016.02.012.
16. Soltanpour Moghadam A, Arabameri M, Baleanu D, Barfeie M. Numerical solution of variable fractional order advection-dispersion equation using Bernoulli wavelet method and new operational matrix of fractional order derivative. Mathematical Methods in the Applied Sciences. 2020;43(7):3936-3953. doi: 10.1002/mma.6164.

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